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Propagators in spherically symmetric backgrounds

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Abstract

We outline a systematic procedure for computing the propagators of quantum fluctuations in spherically symmetric classical backgrounds. As an example, we rederive the propagator for the quantum fluctuations around an instanton background in Yang–Mills theory. The procedure may be applied to several other examples of physical interest, such as fluctuations around vortices, monopoles, skyrmions, etc. Analytic expressions for the propagators may be obtained from analytic solutions of ordinary differential equations. In this construction of the propagator, we present new spherical harmonics for the decomposition $o(4) = su(2) \oplus su(2)$, which are more convenient for the present example.

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1. Introduction

Semiclassical methods are widely used in many areas of physics, either to incorporate quantum fluctuations into classical descriptions or to analyse the classical limit of quantum descriptions [1]. In one-dimensional quantum mechanical examples, semiclassical methods make use of a rather special property of the ordinary differential equations involved: knowledge of the classical solutions of the equations of motion automatically leads to the propagator of fluctuations around them [2]. From the semiclassical propagator, one can construct a whole semiclassical series around each classical solution, either in quantum mechanics [2] or in quantum statistical mechanics, in the analysis of systems in equilibrium with a heat reservoir [3].

The existence of several classical solutions of the equations of motion poses a problem: one has to deal with the interference of the series around solutions that may become 'close' to each other, as they appear or disappear when the parameters of the problem change. Clearly, it is a question of interpolating the subspaces where solutions coalesce, a process which requires going beyond quadratic approximations when traversing the 'caustics' in the appropriate space of parameters that characterizes the solution. Although recipes for performing such interpolations have long been known [4], methods based on catastrophe theory have led to a simplified treatment of problems in quantum statistical mechanics [5, 6].

Higher dimensional quantum mechanics may also benefit from the special property that guarantees the construction of the semiclassical propagator from the knowledge of classical solutions. However, the spectrum of applications where the property can be used is restricted [7]. In quantum field theory, however, either at zero or at finite temperature, the special property does not hold, as one has to treat *partial*, rather than *ordinary*, differential equations. Nevertheless, in cases where classical solutions exhibit special symmetries, exact semiclassical propagators may still be obtained and used to construct the semiclassical series [8, 9].

In the present work, we discuss the construction of semiclassical propagators in QFT in problems with radial symmetry. The study of systems with spherically symmetric potentials encompasses physically relevant scenarios involving instantons, monopoles, vortices, etc. These topological solutions have been shown to play a special role in different theories. In particular, semiclassical expansions around instantons take into account non-perturbative tunnelling effects that are missed in all orders of a perturbative expansion around a trivial vacuum, as in ordinary perturbation theory.

Renewed attention has been devoted to analytic approaches using instanton solutions in QCD. The phenomenological success of instanton-based calculations indicates that they play a role in many features of the strong interactions, such as spontaneous chiral symmetry breaking [10]. Other works consider new effects in the QCD scattering of high-energy partons due to instantons, as well as the suppression of collision-induced tunnelling [11] .

In this paper, we use the well-known method of obtaining multi-dimensional Green's function from the simpler radial Green's functions [12, 13]. As radial is the only relevant direction, we can factorize out the angular dependence, arriving at an essentially tractable one-dimensional problem. We claim that this technique may be useful in non-trivial cases and, as an application, we exhibit a lengthy but straightforward construction of the $SU(2)$ instanton propagator [14]. That propagator was previously obtained in the literature using involved ad hoc methods [15]. Here, we are able to reproduce that quantity using a rather general approach valid over a wide class of potentials with radial dependence. As a by-product, we encountered a new type of generalized spherical harmonics that naturally decouple the $so(4)$ algebra into the sum $su(2) \oplus su(2)$. Such functions can be of use in other problems involving isospin–orbit couplings in four dimensions.

This paper is organized as follows. In section 2, we show how to express the Green's function of an n-dimensional problem with spherical symmetry as a sum over partial radial Green's functions each obeying a given one-dimensional equation. In section 3, we apply that systematic procedure to a non-trivial problem, rederiving the known expression for the scalar instanton propagator in a direct way. Comments on related calculations are given in section 4. Conclusions are summarized in section 5 and explicit calculations used along the paper are found in the appendix.

2. General formulation for spherically symmetric potentials

We consider an n-dimensional Euclidean scalar field theory for $\varphi(x)$, whose actions have the form

$$S[\varphi] = \int \mathrm{d}^n x \left[\frac{1}{2}(\nabla\varphi)^2 + V(\varphi)\right]. \tag{1}$$

The generating functional is

$$Z[j] = \oint [\mathcal{D}\varphi] \exp\left\{-S[\varphi] + \int j\varphi \, \mathrm{d}^n x\right\}. \tag{2}$$

In the semiclassical approximation, $Z[j]$ is dominated by the stationary points of the action. One such classical solution, $\overline{\varphi}$, is assumed to depend only on the radial coordinate r,

$$\left(\frac{\delta S}{\delta \varphi}\right)_{\overline{\varphi}(r)} = -\nabla^2 \overline{\varphi} + V'(\overline{\varphi}) = 0. \tag{3}$$

The first contributions from quantum fluctuations around $\overline{\varphi}$ involve the semiclassical propagator that is defined by

$$[-\nabla^2 + V''(\overline{\varphi})]G(\mathbf{r}, \mathbf{r}') = \delta^n(\mathbf{r} - \mathbf{r}'). \tag{4}$$

Clearly, $V''(\overline{\varphi})$ is a function of a single variable, r. Therefore, we are left with the following Green's function problem:

$$[\nabla^2 + U(r)]G(\mathbf{r}, \mathbf{r}') = \delta^n(\mathbf{r} - \mathbf{r}'). \tag{5}$$

Following [16], we write the Green's function as an n-dimensional partial-wave expansion

$$G(\mathbf{r}, \mathbf{r}') = \sum_{l=0}^{\infty} g_l(r, r') C_l^{\frac{n-2}{2}}(\hat{r} \cdot \hat{r}'), \tag{6}$$

where $C_l^{\frac{n-2}{2}}$ are Gegenbauer polynomials. Restricted to each l-subspace, the n-dimensional Laplace operator reads

$$\nabla_l^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{l(l+n-2)}{r^2}. \tag{7}$$

As is well known, except for $n = 1$, the Laplacian also involves a first derivative. That term will spoil the quantum mechanics result which allows one to express the propagator in terms of the classical solution. However, the problem becomes effectively one dimensional, a simplification which warrants the hopes of deriving exact results in certain cases.

Note that the delta function can be factorized in its radial and angular parts,

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(r - r') \frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n-1}{2}}} \frac{\delta(\hat{r} \cdot \hat{r}' - 1)}{r^{n-1}}. \tag{8}$$

In fact, that expression vanishes whenever $r \neq r'$ and $\hat{r} \cdot \hat{r}' \neq 1$, and integration over the entire space produces one. The Gegenbauer polynomials constitute a complete set on the sphere, i.e.,

$$\sum_{l=0}^{\infty} \sigma_{nl} C_l^{\frac{n-2}{2}}(\hat{r} \cdot \hat{r}') = \delta(\hat{r} \cdot \hat{r}' - 1), \tag{9a}$$

where

$$\sigma_{nl} = \frac{(2l + n - 2)\Gamma^2\left(\frac{n-2}{2}\right)}{2^{4-n}\pi\Gamma(n-2)}. \tag{9b}$$

Therefore, if we were able to solve the equations

$$\left[\nabla_l^2 + U(r)\right] g_l(r, r') = \delta(r - r') \tag{10}$$

(under proper boundary conditions), then the Green's function would be given by

$$G(\mathbf{r},\mathbf{r}') = b_n \sum_{l=0}^{\infty}(2l+n-2)\frac{g_l(r,r')}{(r')^{n+1}} C_l^{\frac{n-2}{2}}(\hat{r}\cdot\hat{r}'), \tag{11a}$$

where

$$b_n = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma^2\left(\frac{n-2}{2}\right)}{2^{5-n}\pi^{\frac{n+1}{2}}\Gamma(n-2)}. \tag{11b}$$

The crucial point here is whether or not the solutions of each of the homogeneous equations (10) are known. Once we know them, we can construct the partial Green's functions g_l using elementary one-dimensional methods. Finally, we use g_l to obtain $G(\mathbf{r},\mathbf{r}')$. Whenever the propagator can be calculated by an ad hoc method, we believe it is possible to reconstruct it using this traditional procedure. We will verify this guess in the case of the scalar propagator in the presence of an instanton background, whose propagator is exactly known [15].

We end this section by working out the (trivial) free case, $U(r) = 0$, as an exercise. Then, the two independent solutions of equation (10) are $\psi_1(r) = r^{-(l+n+2)}$ and $\psi_2(r) = r^l$. In order to build the free partial Green's function, g_{nl}, we need the Wronskian $w(r) = (2l+n+2)/r^{n-1}$. We have

$$g_{nl}(r,r') = \begin{cases} b_1\psi_1(r) + b_2\psi_2(r) - \Omega(r,r'), & r > r', \\ b_1\psi_1(r) + b_2\psi_2(r), & r' > r, \end{cases} \tag{12a}$$

where

$$\Omega(r,r') = \frac{\psi_1(r)\psi_2(r') - \psi_2(r)\psi_1(r')}{w(r')}. \tag{12b}$$

Imposing that g_{nl} be well behaved at $r = 0$ and $r = \infty$, we obtain

$$g_{nl}(r,r') = -(r')^{n-1}\frac{r_<^l}{r_>^{l+n-2}}. \tag{13}$$

Finally, summing over l and using that

$$\frac{1}{(1-2ty+y^2)^\lambda} = \sum_{l=0}^{\infty} C_l^\lambda(t) y^l, \tag{14}$$

we obtain $G^f(\mathbf{r},\mathbf{r}') = -b_n/|\mathbf{r}-\mathbf{r}'|^{n-2}$, with $b_3 = 1/4\pi$ and $b_4 = 1/4\pi^2$, in particular.

In the following, we illustrate the utility of our method by obtaining a closed expression for the instanton scalar propagator.

3. Instanton scalar propagator

In a seminal paper, Belavin, Polyakov, Shvarts and Tyupkin [17] have shown that

$$A_\mu^a(\mathbf{r}) = \frac{2\eta_{a\mu\nu}\mathbf{r}^\nu}{\mathbf{r}^2 + \rho^2} \tag{15}$$

is a minimum of the $SU(2)$ Yang–Mills action with winding number 1. $\eta_{a\mu\nu}$ are called 't Hooft symbols (see appendix A.1). The parameter ρ is arbitrary, the instanton scale, that we will fix to be 1, without loss of generality.

Fluctuations around this stationary point of the action lead to the instanton *scalar* propagator equation

$$-D^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{16}$$

where $D_\mu = \partial_\mu - \mathrm{i}T_a A^a_\mu$ is the covariant derivative and T^a are $SU(2)$ isospin generators. We focus here on the case of isospin 1/2, where T^a is written in terms of Pauli matrices τ^a as $T^a = \tau^a/2$. Cases of higher spin and isospin can be considered in a similar fashion and will be addressed in section 4.

Thus, the D^2 operator is essentially a Laplacian in four dimensions plus terms involving T^2 and the isospin–orbit coupling, with $\mathbf{L}_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$. Following 't Hooft [14], we use $L^2 = -(1/8)L_{\mu\nu}L_{\mu\nu}$ and introduce the spacetime operators

$$M_a = -\frac{\mathrm{i}}{2}\eta_{a\mu\nu}x^\mu\frac{\partial}{\partial x^\nu} \qquad \text{and} \qquad N_a = -\frac{\mathrm{i}}{2}\overline{\eta}_{a\mu\nu}x^\mu\frac{\partial}{\partial x^\nu}, \tag{17}$$

where the symbols $\eta_{a\mu\nu}$ and $\overline{\eta}_{a\mu\nu}$ are defined in appendix A.1. Conversely, we write $L_{\mu\nu} = \mathrm{i}\eta_{a\mu\nu}M^a + i\overline{\eta}_{a\mu\nu}N^a$.

The two sets of operators in (17) obey the $SU(2)$ algebra and commute with each other, realizing the Lie-algebra decomposition $so(4) = su(2) \oplus su(2)$. In [14], 't Hooft uses the suggestive definition[1] $L^2 = -L_{\mu\nu}L_{\mu\nu}/8$, which gives $L^2 = M^2 = N^2$.

Making all the substitutions, we obtain

$$D^2 = \partial_r^2 + \frac{3}{r}\partial_r - \frac{4L^2}{r^2} - \frac{8}{1+r^2}\mathbf{T}\cdot\mathbf{M} - \frac{4r^2}{(1+r^2)^2}T^2. \tag{18}$$

According to our previous considerations, we can think of the coupling $\mathbf{T}\cdot\mathbf{M}$ as an ordinary $SU(2)$ coupling and introduce $\mathbf{J}_1 = \mathbf{M} + \mathbf{T}$. After some manipulation, we obtain

$$\left[\partial_r^2 + \frac{3}{r}\partial_r - \frac{4L^2}{r^2} - \frac{4}{1+r^2}\left(J_1^2 - L^2\right) + \frac{4T^2}{(1+r^2)^2}\right]\psi = 0, \tag{19}$$

or, if ψ is an eigenfunction of J_1^2, L^2 and T^2 (with isospin 1/2),

$$\left[\partial_r^2 + \frac{3}{r}\partial_r - \frac{4\ell(\ell+1)}{r^2} - \frac{4}{1+r^2}(j-\ell)(j+\ell+1) + \frac{3}{(1+r^2)^2}\right]\psi = 0. \tag{20}$$

It is a simple matter to show that equation (20) is of the Morse–Rosen type. Two independent solutions can be obtained for each ℓ in terms of hypergeometric functions. For $\ell = 0$, they are given by [14, 18]

$$\psi_0^{(a)}(r) = \frac{1}{r^2\sqrt{1+r^2}} \qquad \text{and} \qquad \psi_0^{(b)}(r) = \frac{2+r^2}{\sqrt{1+r^2}}, \tag{21}$$

while for $\ell \neq 0$, we have

$$\psi_{j\ell}^{(a)}(r) = \frac{1}{r^{2(\ell+1)}(1+r^2)^{j-\ell}}F_{j\ell}^{(a)}\left(\frac{1}{1+r^2}\right) \tag{22}$$

and

$$\psi_{j\ell}^{(b)}(r) = r^{2\ell}(1+r^2)^{j-\ell}F_{j\ell}^{(b)}\left(\frac{1}{1+r^2}\right), \tag{23}$$

where

$$F_{j\ell}^{(a)}(u) = 1 \qquad \text{and} \qquad F_{j\ell}^{(b)}(u) = 1 + \frac{u}{2\ell+1} \qquad \text{when} \quad j = \ell + 1/2 \tag{24}$$

and

$$F_{j\ell}^{(a)}(u) = 1 - \frac{u}{2\ell+1} \qquad \text{and} \qquad F_{j\ell}^{(b)}(u) = 1 \qquad \text{when} \quad j = \ell - 1/2. \tag{25}$$

The next step is to find a basis of eigenfunctions of J_1^2, L^2, M_3, N_3.

[1] In order to avoid confusion, we will adopt the label ℓ to indicate the momentum angular eigenvalue by 't Hooft and the normal l to indicate the usual eigenvalue. They are related by $\ell = l/2$.

3.1. A simultaneous basis of L^2, M_3 and N_3

The decomposition of the Lie algebra $o(4)$ as $su(2) \oplus su(2)$ is usually presented in textbooks by combining the six angular momentum operators in two independent sets of three operators. This is very suggestive due to its relation with the usual three-dimensional angular momentum algebra. Following this idea, it should be interesting to build a basis of the space of the functions over S^3 composed by eigenfunctions with properties analogous to the usual spherical harmonics, Y_{lm}. We are not aware of it in the classical literature. In fact, the generalized spherical harmonics in S^3 appearing in the literature are given in terms of zonal spherical functions, involving Gegenbauer polynomials. Such a basis is not adaptable to the above decomposition.

It is possible to show that the functions H_{lmn}, defined in terms of the variables $z_1 = x_1 + \mathrm{i}x_2$, $z_2 = x_3 + \mathrm{i}x_4$ and their complex conjugates as

$$H_{\ell,m,n} = \frac{1}{\sqrt{A_{\ell mn}}} z_2^{*m-n} z_1^{\ell+n} z_1^{*\ell-m} {}_2F_1(m-\ell, -n-\ell, m-n+1, -|z_2|^2/|z_1|^2) \\ (m = -\ell, \ldots, \ell; n = -\ell, \ldots, m) \tag{26}$$

and

$$H_{\ell,m,n} = \frac{1}{\sqrt{A_{\ell nm}}} z_2^{n-m} z_1^{\ell+m} z_1^{*\ell-n} {}_2F_1(n-\ell, -m-\ell, n-m+1, -|z_2|^2/|z_1|^2) \\ (n = -\ell, \ldots, \ell; m = -\ell, \ldots, n), \tag{27}$$

where

$$A_{\ell mn} = \frac{(m-n)!^2}{2\ell+1} \frac{(\ell+n)!(\ell-m)!}{(\ell-n)!(\ell+m)!}, \tag{28}$$

form an orthonormal basis of L^2, M_3 and N_3.

Although H_{lmn} are written in terms of hypergeometric functions, they are actually homogeneous polynomials of degree 2ℓ.

All good properties of the functions $H_{\ell mn}$ are what one should expect from the usual Y_{lm}. Therefore, we will accept them and leave the proofs for a future publication.

They satisfy

$$M_3 H_{\ell,m,n} = m H_{\ell,m,n} \qquad \text{and} \qquad N_3 H_{\ell,m,n} = n H_{\ell,m,n}. \tag{29}$$

We also have

$$M_{\pm} H_{\ell,m,n} = \sqrt{\ell(\ell+1) \mp m(m+1)}\, H_{\ell,m\pm1,n} \tag{30}$$

and

$$N_{\pm} H_{\ell,m,n} = \sqrt{\ell(\ell+1) \mp n(n+1)}\, H_{\ell,m,n\pm1}. \tag{31}$$

Furthermore, the set of functions H_{lmn} is complete, i.e.,

$$\sum_{m,n=-\ell}^{\ell} H_{\ell,m,n}(\hat{r}) H^*_{\ell,m,n}(\hat{r}') = (2\ell+1) C^1_{2\ell}(\hat{r}\cdot\hat{r}'). \tag{32}$$

Combining (32) and (9), we obtain

$$\frac{2}{\pi} \sum_{\ell=0}^{\infty} \sum_{m,n=-\ell}^{\ell} H_{\ell,m,n}(\hat{r}) H^*_{\ell,m,n}(\hat{r}') = \delta(\hat{r}\cdot\hat{r}' - 1). \tag{33}$$

3.2. Constructing the propagator

The eigenfunctions of J^2 are doublets with eigenvalues $j = \ell + 1/2$ or $j = \ell - 1/2$ (when $\ell \neq 0$). It is easy to verify that two independent solutions of (19) we need are

$$H^{(a)}_{j,\ell,M,n}(\mathbf{r}) = \frac{\psi^{(a)}_{j\ell}(r)}{\sqrt{2\ell+1}} Y_{j\ell Mn}(\hat{r}) \qquad \text{and} \qquad H^{(b)}_{j,\ell,M,n}(\mathbf{r}) = \frac{\psi^{(b)}_{j\ell}(r)}{\sqrt{2\ell+1}} Y_{j\ell Mn}(\hat{r}), \tag{34}$$

where

$$Y_{j\ell Mn}(\hat{r}) = \begin{pmatrix} \pm\sqrt{\ell \pm M + 1/2}\, H_{\ell,M-1/2,n}(\hat{r}) \\ \sqrt{\ell \mp M + 1/2}\, H_{\ell,M+1/2,n}(\hat{r}) \end{pmatrix}, \qquad \text{for} \quad j = \ell \pm 1/2. \tag{35}$$

From now on, the symbol $\pm$ will correspond to $j = \ell \pm 1/2$. Using one-dimensional methods, we obtain for the radial Green's function

$$g_{j\ell Mn}(\mathbf{r}, \mathbf{r}') = \frac{H^{(a)}_{j,\ell,M,n}(\mathbf{r}_>) \otimes H^{(b)}_{j,\ell,M,n}(\mathbf{r}_<)}{w(r')}. \tag{36}$$

Note that as r goes to infinity, equation (19) reduces to the four-dimensional free equation with J^2 replacing L^2. So, we conclude that the Wronskian is simply $w(r) = 2(2j+1)/r^3$, yielding

$$g_{j\ell Mn}(\mathbf{r}, \mathbf{r}') = \frac{\psi^{(a)}_{j\ell}(r_>)\psi^{(b)}_{j\ell}(r_<)}{2(2j+1)} \frac{1}{2\ell+1} Y_{j\ell Mn}(\hat{r}) \otimes Y_{j\ell Mn}(\hat{r}'). \tag{37}$$

Before going further in the calculation of $g_{j\ell Mn}$, note that

$$\begin{pmatrix} (\ell \pm M + \frac{1}{2}) H_{\ell,M-\frac{1}{2},n}(\hat{r}) H_{\ell,M-\frac{1}{2},n}(\hat{r}') & \sqrt{\ell(\ell+1) - M^2 + \frac{1}{4}} H_{\ell,M-\frac{1}{2},n}(\hat{r}) H_{\ell,M+\frac{1}{2},n}(\hat{r}') \\ \sqrt{\ell(\ell+1) - M^2 + \frac{1}{4}} H_{\ell,M+\frac{1}{2},n}(\hat{r}) H_{\ell,M-\frac{1}{2},n}(\hat{r}') & (\ell \mp M + \frac{1}{2}) H_{\ell,M+\frac{1}{2},n}(\hat{r}) H_{\ell,M+\frac{1}{2},n}(\hat{r}') \end{pmatrix}$$
$$= \begin{pmatrix} \left(\frac{2j+1}{2} \pm M_3\right) H_{\ell,M-\frac{1}{2},n}(\hat{r}) H_{\ell,M-\frac{1}{2},n}(\hat{r}') & \pm M_- H_{\ell,M+\frac{1}{2},n}(\hat{r}) H_{\ell,M+\frac{1}{2},n}(\hat{r}') \\ \pm M_+ H_{\ell,M-\frac{1}{2},n}(\hat{r}) H_{\ell,M-\frac{1}{2},n}(\hat{r}') & \left(\frac{2j+1}{2} \mp M_3\right) H_{\ell,M+\frac{1}{2},n}(\hat{r}) H_{\ell,M+\frac{1}{2},n}(\hat{r}') \end{pmatrix}. \tag{38}$$

Using the completeness relation (32), we arrive at

$$\sum_{M,n} Y_{j\ell Mn}(\hat{r}_>) \otimes Y_{j\ell Mn}(\hat{r}_<) = (2\ell+1) \begin{pmatrix} \frac{2j+1}{2} \pm M_3 & \pm M_- \\ \pm M_+ & \frac{2j+1}{2} \mp M_3 \end{pmatrix} C^1_{2\ell}(\hat{r} \cdot \hat{r}'). \tag{39}$$

Inserting (39) into an isospin-adapted form of equation (11a), we obtain

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}') &= \frac{1}{2\pi^2} \sum_{\ell=0}^{\infty} \sum_{\pm} \sum_{M,n} g_{j\ell Mn}(\mathbf{r}, \mathbf{r}') \\ &= \frac{1}{4\pi^2} \sum_{\ell=0}^{\infty} \sum_{\pm} \frac{\psi^{(a)}_{j\ell}(r_>)\psi^{(b)}_{j\ell}(r_<)}{2j+1} \begin{pmatrix} \frac{2j+1}{2} \pm M_3 & \pm M_- \\ \pm M_+ & \frac{2j+1}{2} \mp M_3 \end{pmatrix} C^1_{2\ell}(\hat{r} \cdot \hat{r}') \\ &= \frac{1}{4\pi^2} \sum_{\ell=0}^{\infty} \sum_{\pm} \left[\frac{1}{2} \pm \frac{1}{2j+1} \begin{pmatrix} M_3 & M_- \\ M_+ & -M_3 \end{pmatrix} \right] \frac{\psi^{(a)}_{j\ell}(r_>)\psi^{(b)}_{j\ell}(r_<)}{2} C^1_{2\ell}(\hat{r} \cdot \hat{r}') \\ &= \frac{1}{4\pi^2} \sum_{\ell=0}^{\infty} \sum_{\pm} \left[\frac{1}{2} \pm \frac{M_a \tau_a}{2j+1} \right] \frac{\psi^{(a)}_{j\ell}(r_>)\psi^{(b)}_{j\ell}(r_<)}{2} C^1_{2\ell}(\hat{r} \cdot \hat{r}'). \end{aligned} \tag{40}$$

In appendix A.1, we show that

$$(M_a\tau_a)C^1_{2\ell}(\hat{r}\cdot\hat{r}') = -\mathrm{i}\eta_{\alpha\mu\nu}\frac{r_\mu r'_\nu}{rr'}\tau_\alpha C^2_{2\ell-1}(\hat{r}\cdot\hat{r}'). \tag{41}$$

Then,

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi^2}\sum_{\ell=0}^{\infty}\sum_{\pm}\psi^{(a)}_{j\ell}(r_>)\psi^{(b)}_{j\ell}(r_<)\left[\frac{C^1_{2\ell}(\hat{r}\cdot\hat{r}')}{2} \mp \frac{\mathrm{i}}{rr'}\eta_{\mu\nu\alpha}x_\mu y_\nu\tau_\alpha\frac{C^2_{2\ell-1}(\hat{r}\cdot\hat{r}')}{2j+1}\right]. \tag{42}$$

So far, we have not used the expression for $\psi^{(a)}_{j\ell}$ and $\psi^{(b)}_{j\ell}$. Substituting their expressions into (42), we finally obtain, after a tedious but simple algebraic calculation (see appendix A.2),

$$G(\mathbf{r},\mathbf{r}') = \frac{1+\mathbf{r}\cdot\mathbf{r}' + \mathrm{i}\eta_{\alpha\mu\nu}r_\mu r'_\nu\tau_\alpha}{4\pi^2|r-r'|^2\sqrt{(1+r^2)(1+r'^2)}}, \tag{43}$$

which coincides with the expression known in the literature [15].

4. Handling related instanton propagators

Throughout this paper, we have concentrated on the study of the scalar propagator in the presence of an instanton for a massless case with isospin $1/2$. However, the present method applies as well to variations of that problem. In fact, solutions of equation (19) for the massless case with higher isospin have the same structure for ψ_{jl}, i.e., they are written as a product of a simple function of r times a polynomial in $(1+r^2)^{-1}$, allowing for an exact calculation of the propagator. For each value of T^2, we would have a different treatment of the angular part in order to arrive at an expression similar to (42). However, for the remainder of the calculation, we would proceed as in our presentation for the $1/2$ case.

The massive case is more complicated, but also tractable in principle. Using the full expression for the massless semiclassical propagator, one can implement an expansion for the massive propagator in powers of m^2, as found in [19, 20]. Alternatively, one can take approximate solutions of the corresponding one-dimensional radial equation and plug them in (42) in a regular partial-wave expansion, but with the isospin part already simplified. In the particular case of the space-dependent mass introduced by 't Hooft [14], exact solutions are given by (22) and (23), where $F^{(a)}_{jl}$ and $F^{(b)}_{jl}$ are then well-defined hypergeometric functions whose series do not terminate, probably spoiling the possibility of obtaining a simple analytic result, however providing again approximate expressions.

Finally, it is possible to show [15] that for the massless case the propagator of fermions, ghosts and vectors can be written in terms of the scalar propagator, equation (43). In reality, what will be related to the scalar case are the parts of their propagators orthogonal to the zero-mode subspaces. Reference [21] handles the discussion of the zero modes. If one uses a certain parametrization of the instanton background in terms of a radial scalar function ϕ given by a kink,

$$\phi(t) = \tanh(t-t_s), \tag{44}$$

one ends up with an operator of the Morse–Rosen type. The existence of zero modes in the original problem is equivalent to the existence of bound states for the Morse–Rosen associated problem whose eigenfunctions are completely known [21].

5. Conclusions

We have derived a systematic procedure for obtaining a semiclassical propagator for the case of spherically symmetric backgrounds. It provides a step-by-step construction of the scalar instanton propagator, previously obtained by using ad hoc methods. Having a construction which is easily related to more traditional methods is not only desired, but may also prove useful in extensions to other cases.

As a by-product of our procedure, we were led to spherical harmonics which are very suitable to the decomposition of $SO(4)$ into a direct sum, $su(2) \oplus su(2)$. This is of great utility in problems that involve the coupling of two such $su(2)$ s, as in the case of isospin–orbit or spin–isospin orbit couplings frequently encountered in the literature.

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Appendix. Details of the calculation

A.1. Equation (41)

In order to arrive at equation (41), first note that $L_{ij}(\hat{r}\cdot\hat{r}') = L_{ij}(\mathbf{r}\cdot\mathbf{r}')/(rr')$. From definition (17), we see that

$$\begin{aligned} M_a\tau_a(\mathbf{r}\cdot\mathbf{r}') &= -\frac{\mathrm{i}}{2}\eta_{a\mu\nu}x_\mu\partial_\nu(x_\sigma x'_\sigma)\tau_a \\ &= -\frac{\mathrm{i}}{2}\eta_{a\mu\nu}x_\mu x'_\nu\tau_a, \end{aligned}$$

where $\eta_{a\mu\nu}$ and $\overline{\eta}_{a\mu\nu}$ are totally antisymmetric quantities satisfying

$$\begin{aligned} &\eta_{a\mu\nu} = \epsilon_{a\mu\nu}, \qquad \text{for} \quad \mu,\nu = 1,2,3, \\ &\eta_{a4\nu} = -\delta_{a\nu}, \qquad \eta_{a\mu4} = \delta_{a\mu}, \qquad \overline{\eta}_{a\mu\nu} = (-1)^{\delta_{\mu4}+\delta_{\nu4}}\eta_{a\mu\nu}. \end{aligned}$$

To complete the proof, one needs to use the following property of the Gegenbauer polynomials [16]:

$$\frac{\mathrm{d}C_l^1(t)}{\mathrm{d}t} = 2C_{l-1}^2(t).$$

A.2. Final steps

In order to obtain expression (43), we note that, after summing over j, the term proportional to the identity in (42) becomes

$$\begin{aligned} &\frac{1}{8\pi^2}\left[\left(\sqrt{\frac{1+r_>^2}{1+r_<^2}}+\sqrt{\frac{1+r_<^2}{1+r_>^2}}\right)\frac{1}{r_>^2}\sum_{l=1}^{\infty}\left(\frac{r_<}{r_>}\right)^l C_l^1(\hat{r}\cdot\hat{r}') + \frac{2+r_<^2}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}r_>^2}\right] \\ &\qquad = \frac{1}{8\pi^2}\left[\frac{2+r_<^2+r_>^2}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\left(\frac{1}{|r-r'|^2}-\frac{1}{r_>^2}\right)+\frac{2+r_<^2}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}r_>^2}\right] \\ &\qquad = \frac{1}{8\pi^2}\left[\frac{2+r_<^2+r_>^2}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\frac{1}{|r-r'|^2}-\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\right] \\ &\qquad = \frac{1}{4\pi^2}\frac{1+\mathbf{r}\cdot\mathbf{r}'}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\frac{1}{|r-r'|^2}. \end{aligned}$$

Similarly, the term in τ_α is given by

$$
\begin{aligned}
&\frac{\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2(rr')}\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}r_>^2}\sum_{l=1}^{\infty}\left[\left(1-\frac{1}{(l+1)(1+r_>^2)}\right)\frac{(1+r_>^2)}{l}\right.\\
&\qquad\left.-\left(1+\frac{1}{(l+1)(1+r_<^2)}\right)\frac{(1+r_<^2)}{l+2}\right]\left(\frac{r_<}{r_>}\right)^l C_{l-1}^2(\hat r\cdot\hat r')\\
&\quad=\frac{\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2(rr')}\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}r_>^2}\sum_{l=1}^{\infty}\left[\frac{r_>^2}{l}-\frac{r_<^2}{l+2}\right]\left(\frac{r_<}{r_>}\right)^l C_{l-1}^2(\hat r\cdot\hat r')\\
&\quad=\frac{\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2(rr')}\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\sum_{l=1}^{\infty}\left[\frac{1}{l}\left(\frac{r_<}{r_>}\right)^l-\frac{1}{l+2}\left(\frac{r_<}{r_>}\right)^{l+2}\right]C_{l-1}^2(\hat r\cdot\hat r')\\
&\quad=\frac{\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2(rr')}\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\left[\sum_{l=3}^{\infty}\left(\frac{r_<}{r_>}\right)^l\frac{1}{l}\left(C_{l-1}^2(\hat r\cdot\hat r')-C_{l-3}^2(\hat r\cdot\hat r')\right)\right.\\
&\qquad\left.+\frac{r_<}{r_>}+2\left(\frac{r_<}{r_>}\right)^2\hat r\cdot\hat r'\right]\\
&\quad=\frac{\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2(rr')}\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\left[\sum_{l=3}^{\infty}\left(\frac{r_<}{r_>}\right)^l\left(C_{l-1}^2(\hat r\cdot\hat r')+C_{l-3}^2(\hat r\cdot\hat r')\right)\right.\\
&\qquad\left.-2(\hat r\cdot\hat r')C_{l-2}^2(\hat r\cdot\hat r')+\frac{r_<}{r_>}+2\left(\frac{r_<}{r_>}\right)^2\hat r\cdot\hat r'\right]\\
&\quad=\frac{\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2}\frac{1}{\sqrt{1+r_<^2}\sqrt{1+r_>^2}}\frac{1}{|r-r'|^2}.
\end{aligned}
$$

The previous result used the relation [18]

$$(l+2)C_{l+2}^p(t)=2(p+l+1)tC_{l+1}^p-(2p+l)C_l^p(t),$$

as well as the sequence of identities below:

$$
\begin{aligned}
\sum_{l=2}^{\infty}\left(\frac{r_<}{r_>}\right)^l C_l^2(\hat r\cdot\hat r')&=r_<r_>^3\frac{1}{|r-r'|^4}-\frac{r_<}{r_>}\left(1+4\frac{r_<}{r_>}(\hat r\cdot\hat r')\right),\\
\sum_{l=1}^{\infty}\left(\frac{r_<}{r_>}\right)^l C_l^2(\hat r\cdot\hat r')&=r_<^2r_>^2\frac{1}{|r-r'|^4}-\left(\frac{r_<}{r_>}\right)^2,\\
\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l C_l^2(\hat r\cdot\hat r')&=r_<^3r_>\frac{1}{|r-r'|^4}.
\end{aligned}
$$

Finally, we obtain the desired expression for the propagator:

$$G(\mathbf{r},\mathbf{r}')=\frac{1+\mathbf{r}\cdot\mathbf{r}'+\mathrm{i}\eta_{\mu\nu\alpha}r_\mu r'_\nu\tau_\alpha}{4\pi^2|r-r'|^2\sqrt{(1+r^2)(1+r'^2)}}.$$

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